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A study of the dynamical relaxation of unstable states driven by Gaussian exponentially correlated noise

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Abstract. The coloured-noise (Ornstein–Uhlembeck process) problem corresponding to the decay process of unstable states, is studied using the nonlinear relaxation time and the quasideterministic approach. The effect of the non-Markovian character in this problem as well as the statistics of the initial conditions are both addressed. Our general result is applied to the Landau model as a typical example.

1. Introduction

In the last few years much attention has been devoted to the study of the time decay of unstable states. The nonlinear relaxation time (NLRT) has itself been proposed as a characterization of the time scale associated with such decay process.

In [1] the behaviour of the relaxation time scale for unstable states induced by Gaussian white noise in the context of the NLRT and the quasideterministic (QD) theory [2] is widely discussed. There, it is shown that the universal character of this time scale and the characteristics of the types of models appear in a natural way. The study of the corresponding Gaussian coloured noise (GCN) represents a very interesting case because it is not ideal but a noise which exhibits a finite correlation time which can be used to model real situations. In this paper, the same framework theory is applied in studying the decay of unstable states driven by GCN, although this case has also been analysed by other theoretical mechanisms. In fact, [3, 4] gives an account of the general formulation of the NLRT for describing the decay of unstable states driven by GCN. However, this study is made in terms of two Markovian variables and the corresponding Fokker–Planck equation; the method is very complicated and, besides, it leads to an approximate result for the NLRT.

In this work we basically intend to show that the theoretical treatment based on the QD approach and the NLRT to characterize the dynamical relaxation of unstable states is simpler than the method proposed in [3, 4]. In this context, the time scale can be expressed in terms of the finite correlation time τ without any approximation in this parameter and of the nonlinear contributions of any unstable model. The study of the QD approach shows that the correlation between the noise and system variables at the initial stage arises in a natural way. Here we follow an idea suggested in [5] which reports the results for a linear stochastic model in terms of the mean-passage times and compares them with results obtained by Suzuki [6] and by analogue simulation [7]. In this work we will study the case in which the initial state of the system is physically determined by the same noise responsible for the decay process and the situations in which the system and noise are considered as being statistically independent at the initial state [3, 4]. The structure of this paper is as follows. In section 2, we define the NLRT and study its connection with the QD approach. We study the QD theory and the effect of the initial conditions as well as the non-Markovian contributions on the problem. A general expression for the NLRT is obtained for any unstable system driven by GCN, in the limit of small noise intensity. In section 3, the Landau model is studied as a typical example. The approximations for small correlation are specifically analysed in order to compare them with those obtained in [3, 4]. Concluding remarks are given in section 4.

2. The NLRT and QD approach with GCN

2.1. The NLRT

We define the NLRT as in [1, 3, 4]. In this definition the physical quantity, for which the relaxation from an initial state to the corresponding steady state is to be studied, has to be specified. In this case, such a variable will be the average of the modulus r of the physical quantity, i.e. $\langle r\rangle = \langle x^2\rangle$. Then, the NLRT reads

$$T = \int_0^\infty \frac{\langle r(t)\rangle - \langle r\rangle_{\rm st}}{\langle r(0)\rangle - \langle r\rangle_{\rm st}}\,{\rm d}t. \tag{2.1}$$

Let us define $M_0 = \langle r(0)\rangle - \langle r\rangle_{\rm st}$. The QD analysis considers the fluctuations around the unstable initial state as the mechanism responsible for initiating the relaxation. In this case, the initial point $\langle r(0)\rangle$ is assumed to be a stochastic variable, called h, which takes the initial fluctuations of the system into account. The connection between equation (2.1) and the QD approach is essentially based on the deterministic evolution of the modulus r given by the equation

$$\dot{r} = v(r) \tag{2.2}$$

where $v(r)$ is given by [1]

$$v(r) = \frac{r(r_{\rm st} - r)}{C_0 + rg(r)} \tag{2.3}$$

which defines, in general, a function with an unstable point at $r = 0$, and a stable point at $r = r_{\rm st}$. Here $C_0 = r_{\rm st}/2a$ is a constant and $g(r) \geqslant 0$ is a polynomial of the form $g(r) = \Sigma_{n=0}^{l} a_n r^n$.

Under these circumstances the NLRT (2.1) can be written as

$$T = \frac{1}{M_0}\left\langle \int_{r(0)=h^2}^{r_{\rm st}} (r - r_{\rm st})\frac{{\rm d}r}{v(r)}\right\rangle. \tag{2.4}$$

Direct substitution of (2.3) into (2.4) gives

$$T = -\frac{C_0}{M_0}\left\langle \ln\left[\frac{r_{\rm st}}{h^2}\right]\right\rangle - \frac{1}{M_0}\left\langle \int_{r(0)=h^2}^{r_{\rm st}} g(r)\,{\rm d}r\right\rangle. \tag{2.5}$$

The first logarithmic term is a universal one which governs the decay time of the unstable states. It arises from the study of NLRT (2.1) of the deterministic linear model $\dot{r} = 2ar$, as suggested in [1]. Then the NLRT for this linear model is

$$T_{\rm L} = -\frac{C_0}{M_0}\left\langle \ln\left[\frac{r_{\rm st}}{h^2}\right]\right\rangle - C \tag{2.6}$$

where C is a constant given by $C = (1/2a\ M_0)[\langle h^2\rangle - r_{st}]$. Therefore, the time scale (2.5) can be written as

$$T = T_L + C - \frac{1}{M_0}\left\langle \int_{r(0)=h^2}^{r_{st}} g(r)\,\mathrm{d}r \right\rangle. \tag{2.7}$$

We can observe from (2.3) that, depending on the type of unstable model, $g(r)$ can be equal to zero or not. In the case in which $g(r)$ is not equal to zero, it will take into account the nonlinear terms of the model. Therefore, the integral term in the time scale (2.7) accounts for the nonlinearities of the system and noise plays no essential role. The stochastic character of the problem is strictly contained in the terms $\langle \ln h^2\rangle$ and $\langle h^2\rangle$ of T_L, the statistical properties of which will be determined from QD theory.

2.2. *The QD approach with GCN*

Because of the characteristics of the QD analysis, it will be sufficient to study the transient dynamics of the unstable states in terms of the linear Langevin equation for the variable x; i.e. [1]

$$\dot{x} = ax + \xi(t) \tag{2.8}$$

where $a > 0$ and $\xi(t)$ is the stochastic force or noise which is assumed to be Gaussian with zero mean and an Ornstein–Uhlembeck correlation function

$$\langle \xi(t)\xi(t')\rangle = \frac{D}{\tau}\exp\left(-\frac{|t-t'|}{\tau}\right) \tag{2.9}$$

where D is the noise intensity and τ is the correlation time.

The general solution of (2.8) gives

$$x(t) = h(t)\mathrm{e}^{at} \tag{2.10}$$

where

$$h(t) = x_0 + \int_0^t \xi(t')\mathrm{e}^{-at'}\,\mathrm{d}t' \tag{2.11}$$

and $x_0 = x(0)$ is the initial value of the dynamical variable x. Our next step is to show that the process $h(t)$ will play the role of an effective initial condition for long times.

Then the second moment of $h(t)$ can be written in a formal way as

$$\langle h^2(t)\rangle = \langle x^2\rangle_0 + \int_0^t\int_0^t \langle \xi(s')\xi(s)\rangle \mathrm{e}^{-a(s'+s)}\,\mathrm{d}s'\,\mathrm{d}s + 2\int_0^t \langle x(0)\xi(s')\rangle \mathrm{e}^{-as'}\,\mathrm{d}s' \tag{2.12}$$

where the last term shows a dependence on the correlation between the noise ξ and system variables x at time $t = 0$. This means, in general, that in the initial state of the dynamical relaxation both variables are statistically dependent. On the other hand, equation (2.12) clearly shows that there are three interesting cases concerning the behaviour of the initial correlation between the variables (x, ξ).

(i) First, the general case is considered when the initial condition $x(0)$ is assumed to be initially distributed around the unstable state and coupled with the noise [3–5]. This means that $\langle x(0)\xi(t)\rangle \neq 0$. This case can be seen in the following physical situation. At time $t = 0$, the system is in a stable state associated with the value of the control parameter $a = -a_0$. Then, for $t < 0$, the system is described by a linear approximation by the dynamics

$$\dot{x} = -a_0 x + \xi(t). \tag{2.13}$$

The solution of equation (2.13) leads to

$$x_0 = x(0) = \int_{-\infty}^{0} \xi(t') e^{a_0 t'} \, dt'. \tag{2.14}$$

At time $t = 0$, the control parameter is instantaneously changed from a_0 to the value $a > 0$ and the system becomes unstable. The exact $P(x, t)$ of the dynamics (2.13) has already been calculated in [8]. In particular, the stationary distribution is Gaussian and has a second moment given by

$$\langle x^2 \rangle_0 = \frac{D}{a_0(1 + a_0\tau)}. \tag{2.15}$$

Here, the effect of the coloured noise with respect to the white noise ($\tau = 0$) is the renormalization of the noise intensity D by the factor $1/a_0(1 + a_0\tau)$. With these characteristics for x_0, the process $h(t)$ given in equation (2.12) is then a Gaussian process with zero mean and a second moment given by

$$\langle h^2(t) \rangle = \langle x_0^2 \rangle + \frac{D}{a(1 + a\tau)} \left[1 - e^{-2at} + \frac{2a\tau}{(1 - a\tau)} (e^{-(a+\tau^{-1})t} - e^{-2at}) \right] + \frac{2D\tau}{(1 + a_0\tau)(1 + a\tau)} (1 - e^{-(a+\tau^{-1})t}) \tag{2.16}$$

where the last term accounts for the coupling between x_0 and $\xi(t)$ at $t > 0$. So, for times $at \gg 1$, the process $h(t)$ can be approximated by a Gaussian random variable $h(\infty) = h$ with zero mean and variance

$$\sigma_c^2 = \langle h^2 \rangle = \langle x_0^2 \rangle + \frac{D}{a(1 + a\tau)} + \frac{2D\tau}{(1 + a_0\tau)(1 + a\tau)} \tag{2.17}$$

where the subscript c denotes the coupled case.

(ii) The second example corresponds to the decoupled case which is considered when the initial condition $x(0)$ is again initially distributed, but the distribution is independent of the noise. Mathematically, we mean that $\langle x(0)\xi(t) \rangle = 0$. It is also clear from equation (2.12) or (2.16) that only the first two terms survive and that for long times $at \gg 1$, the process $h(t)$ becomes a Gaussian random variable with zero mean and variance

$$\sigma_d^2 = \langle h^2 \rangle = \langle x_0^2 \rangle + \frac{D}{a(1 + a\tau)}. \tag{2.18}$$

Subscript d refers to the decoupled case. Physically, case (ii) means that the noise sources for $t < 0$ and $t > 0$ are not the same or the system starts from a point distributed at random.

(iii) The third case assumes that the initial condition is not initially distributed; instead it is fixed at the unstable state, i.e. $x(0) = 0$, and there is no correlation with the noise. Here, the only term in equation (2.12) or (2.16) which is not zero is the second one. Therefore, the process $h(t)$ plays the role of a stochastic initial condition because, for $at \gg 1$, $h(t)$ can be approximated by a Gaussian random variable with zero mean and variance

$$\sigma_{\mathrm{f}}^2 = \langle h^2 \rangle = \frac{D}{a(1+a\tau)}. \tag{2.19}$$

Here the subscript f denotes the fixed initial condition. This result shows that the effect of the noise correlation time modifies the variance of the effective random initial condition and the noise intensity D has been renormalized to the value $D/a(1+a\tau)$.

From equations (2.17)–(2.19), we can conclude that the probability distribution for the Gaussian random variable h has the following structure

$$P(h) = \frac{2\alpha}{\sqrt{\pi}} \mathrm{e}^{-\alpha^2 h^2} \tag{2.20}$$

where $\alpha^2 = 1/2\sigma^2$.

2.3. The general expression of the NLRT for small noise intensity

We can calculate the NLRT (2.7) for any nonlinear unstable system by first evaluating its linear approximation (2.6). The constants M_0 and C can be calculated as follows. If $\langle r(0) \rangle = 0$ (fixed initial condition) then it is clear that $M_0 = -r_{\mathrm{st}}$. However, if $\langle r(0) \rangle$ is not equal to zero, we can assume that it is proportional to D according to equation (2.15). Now, the second term of M_0 can be obtained using the P_{st} associated with the linear stochastic dynamics of the variables (x, ξ) [3, 4]. In this case, P_{st} is proportional to $\exp\{ax^2/2D/(1-a\tau)\}$, and then the average $\langle r \rangle_{\mathrm{st}}$ will be proportional to $r_{\mathrm{st}} - D$. Therefore, once again in the limit of small noise intensity D, $M_0 \approx -r_{\mathrm{st}}$.

We can evaluate the average $\langle h^2 \rangle$ given in the constant C using the marginal probability (2.20). We find that $\langle h^2 \rangle = 1/2\alpha^2$ which is proportional to σ^2 and, at the same time, proportional to D as can be seen in equations (2.17)–(2.19). This term is also neglected for small noise intensity. Finally, the quantity $\langle \ln h^2 \rangle$ in (2.6), according to (2.20), reduces to $\langle \ln h^2 \rangle = -\ln \alpha^2 + \Psi(1/2)$ where $\Psi(x)$ is the digamma function [9] and α^2 is defined as before.

Therefore, in the framework of our theory, we can characterize the decay of the unstable systems driven by GCN by a time scale associated with the quantity $\langle r \rangle$ that in the limit of small noise intensity takes the most general expression according to equations (2.6) and (2.7)

$$T = \frac{1}{2a} \ln\left(\frac{1}{2D}\right) + B(\tau) + B_0 + \frac{1}{r_{\mathrm{st}}} \int_{r(0)=h^2}^{r_{\mathrm{st}}} g(r)\,\mathrm{d}r + \mathcal{O}(D) \tag{2.21}$$

and the time scale for the linear model is

$$T_{\mathrm{L}} = \frac{1}{2a} \ln\left(\frac{1}{D}\right) + B(\tau) + B_0 + \mathcal{O}(D). \tag{2.22}$$

The first term in equation (2.21) contains the dominant contribution for small noise intensity D which is independent of the correlation time τ. The term $B(\tau)$ contains the non-Markovian contribution as well as the type of coupling between the noise and system. B_0 is a typical constant characteristic of the decay process. Finally, the nonlinear contributions appear obviously in the integral term of (2.21). So, from our general result, three specific time scales for finite correlation time τ can be obtained according to the effect of the initial conditions on the system. Each time scale has its own corresponding $B(\tau)$ and B_0.

When the initial coupling between the noise and the system (i) is taken into account, the second term of (2.21) can be written, with the help of (2.17), as

$$B(\tau) = \frac{1}{2a} \ln \left[\frac{(1+a\tau)(1+a_0\tau)}{1+(a+a_0)\tau} \right] \tag{2.23}$$

and

$$B_0 = \frac{1}{2a} \left\{ \ln \left[\frac{2r_{st}e^{\gamma}}{a^{-1}+a_0^{-1}} \right] - \frac{1}{2} \right\}. \tag{2.24}$$

On the other hand, if the initial state is distributed around the unstable state but decoupled with the noise (ii), then the value of $B(\tau)$, calculated using equation (2.18), is

$$B(\tau) = \frac{1}{2a} \ln \left[\frac{(1+a\tau)(1+a_0\tau)}{1+(a^2+a_0^2)/(a_0+a)\tau} \right] \tag{2.25}$$

and B_0 is the same as (2.24).

The simplest case under consideration is when both variables are statistically independent and the initial condition is fixed (iii). Here we use (2.19) to obtain

$$B(\tau) = \frac{1}{2a} \ln(1+a\tau) \tag{2.26}$$

and

$$B_0 = \frac{1}{2a} \ln \left[2ar_{st}e^{\gamma} - \frac{1}{2} \right]. \tag{2.27}$$

The time scale (2.22) can be compared with those obtained in [5, 6] because the authors analysed only the linear model. In this case, their results are the same as (2.22)–(2.24) except that the constant $(\frac{1}{2})$ which appears in B_0 is characteristic of the NLRT [1, 3, 4].

3. The Landau model

We are now going to study the Landau model as a typical example of a nonlinear system. We will show that the characteristic times obtained in [3] can be reproduced in the limit of small correlation time. The Landau model reads

$$\dot{x} = ax - bx^3 + \xi(t). \tag{3.1}$$

The associated deterministic equation for the modulus $r = x^2$ can be written as

$$\dot{r} = 2ar - 2br^2 \tag{3.2}$$

where, in this case, the function $v(r)$ of (2.3) is $v(r) = 2ar - 2br^2$. Therefore, $r_{st} = a/b$ and $g(r) = 0$. Then, the integral term of (2.21) disappears. In consequence, the NLRT for this model will be

$$T = \frac{1}{2a} \ln \left(\frac{1}{D} \right) + B(\tau) + B_0 + \mathcal{O}(D) \tag{3.3}$$

where $B(\tau)$ and B_0 will be given according to the cases analysed above.

In the limit of small correlation time we use the approximation $\ln(1+x) \approx x$ so that: for (i), $B(\tau)$ is neglected and equation (3.3) reduces to

$$T^{\mathrm{C}} = \frac{1}{2a} \ln\left[\frac{r_{\mathrm{st}} a_0 a}{2D(a_0+a)}\right] + \frac{1}{2a}\left(\gamma + 2\ln 2 - \frac{1}{2}\right) + O(D); \tag{3.4}$$

in (ii), $B(\tau) \approx a_0\tau/(a_0+a)$ and for the decoupled case we get

$$T^{\mathrm{D}} = T^{\mathrm{C}} + \left(\frac{\tau}{1+a/a_0}\right) + O(D); \tag{3.5}$$

and finally, for (iii), $B(\tau) \approx \tau/2$ and the time scale reads

$$T^{\mathrm{F}} = \frac{1}{2a} \ln\left[\frac{a r_{\mathrm{st}}}{2D}\right] + \frac{1}{2a}\left(\gamma + 2\ln 2 - \frac{1}{2}\right) + \frac{\tau}{2} + O(D). \tag{3.6}$$

The equations (3.4)–(3.6) are exactly the same as those obtained in [3] for the limit of small noise intensity and small correlation time. In [4] the comparison with analogue experiments and digital simulations of the NLRT is reported. The results obtained are found to be in excellent agreement with the theoretical predictions of [3] which in this case are the same as those obtained at the end of this section.

4. Concluding remarks

In this paper we have established the general result for the time scale (2.21) of the decay process of an unstable state driven by Gaussian coloured noise. This result is more general than that obtained in [5] because it contains the nonlinear contributions of any unstable model. Our result is exact for finite correlation time, in contrast with the result of [3] in which the NLRT is reported in terms of small τ. Again, the universal character of the logarithmic term is exhibited and corresponds to the early stages of the decay of the unstable state which is dominated by the noise and linear terms. The non-Markovian contribution is clearly contained only in the second term $B(\tau)$ of (2.21) which accounts for the type of coupling between the system and noise at time $t = 0$.

The analogue experiments and digital simulation of the NLRT are reported in [4] with excellent agreement with the theoretical results in the limit of small noise and small correlation time. In those limits, the theoretical results are the same as those obtained in this work.

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